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On Identities of Baric Algebras and Superalgebras

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1. INTRODUCTION

Polynomial identities are important tools in the structure theory of algebras. Finiteness conditions written in the form of identities are invariant under basic algebraic constructions (direct sums, subalgebras, and homomorphic images) and, therefore, are effective in various classes, including genetics algebras [4, 5, 8]. As far as such algebras arise in connection with a certain mathematical model in population genetics, their identities must have a genetic meaning. In this work we study this correspondence and solve some problems in this way.

First of all, we recall some basic notions (cf. [18]). Consider an infinitely large random mating population which produced n genetically distinct gametes $a_1, \ldots, a_n$. Then γ_{ijk}, $i, j, k = 1, \ldots, n$, are the segregation coefficients (γ_{ijk} is the probability that a zygote $a_i a_j$ produces a gamete a_k). Note that for any i, j, k, $\gamma_{ijk} = \gamma_{jik}$, $0 \le \gamma_{ijk} \le 1$ and $\sum_{k=1}^{n} \gamma_{ijk} = 1$. A set P of populations where the number of genetically distinct gametes and segregation coefficients is fixed, is said to be *a population class*. Then a

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population in P is described by a vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{R}^n$ of the gamete frequencies $x_i \geq 0$, $\sum_{i=1}^n x_i = 1$. The set of such vectors is the $n - 1$-dimensional simplex $\Delta = \Delta_P \subset \mathbf{R}^n$.

Take two points $\mathbf{x}, \mathbf{x}' \in \Delta$. The probability that a pair of gametes randomly chosen in the populations $\mathbf{x}, \mathbf{x}'$ produces a gamete a_k is equal to $(\mathbf{x} \cdot \mathbf{x}')_k = \sum_{i,j=1}^n x_i x_j' \gamma_{ijk}$. This defines a multiplication on the simplex Δ, which can be extended to the vector space $A = \mathbf{R}^n$. Thus, we have obtained a stochastic algebra (A, Δ). Note that such an algebra is commutative.

Observe that a convex combination $\lambda \mathbf{x} + (1 - \lambda)\mathbf{x}'$, where $1 \geq \lambda \geq 0$, of two points $\mathbf{x}, \mathbf{x}' \in \Delta$ has genetic meaning (cf. [18, p. 1]). Therefore, a polynomial $f(\mathbf{x^1}, \dots, \mathbf{x^n})$ constructed by means of these operations (convex combinations and products of populations) from given $\mathbf{x^1}, \dots, \mathbf{x^n} \in \Delta$ describes a way of evolution of the populations $\mathbf{x^1}, \dots, \mathbf{x^n}$. Hence, an evolution law of the population class P is an equality of two such polynomials which holds for any $\mathbf{x^1}, \dots, \mathbf{x^n} \in \Delta$.

For example, a population class with the simple Mendelian inheritance [18, p. 8] satisfies the identity

$$x^2 y = xy. \tag{1}$$

Evidently, this implies the identity

$$(x^2)^2 = x^2 \tag{2}$$

which defines the class of Bernstein populations [18, p. 203].

Now we are going to fix some basic notations and to define an evolution law in a precise form.

Define the category of stochastic algebras $\mathcal{S}$, where objects are stochastic algebras, a morphism from (A, Δ) to (A', Δ') is a homomorphism of the algebra A to A' which sends Δ to Δ'. As a matter of fact, the condition of a finite dimension is unnecessary. Thus, from this point of view the free commutative algebra $\mathbf{R}\langle X \rangle$ with a set of free generators X is a stochastic algebra. Actually, $\mathbf{R}\langle X \rangle$ is the monoid algebra of the free commutative (nonassociative) monoid $V\langle X \rangle$ (i.e., the set of commutative monomials in X is a basis of $\mathbf{R}\langle X \rangle$ over $\mathbf{R}$), hence, the simplex of states $\Delta\langle X \rangle$ is the set of all convex combinations of elements in $V\langle X \rangle$. It is clear that $(\mathbf{R}\langle X \rangle, \Delta\langle X \rangle)$ is a free object in $\mathcal{S}$. Also, we may characterize it in the following way.

PROPOSITION 1. *The stochastic algebra* $(\mathbf{R}\langle X \rangle, \Delta\langle X \rangle)$ *can be represented as an inverse limit of a family of finite dimensional stochastic algebras.*

Proof. Consider the subset $U_{n,m}$ of all monomials in $V\langle X \rangle$ which depend only on $x_1, \dots, x_n$ and have degree less than or equal to m.

Further, denote

$$I_{n,m} = \mathbf{R}\{u - x_{n+1} | u \in V\langle X \rangle, u \notin U_{n,m}\},$$

that is, $I_{n,m}$ is the linear span of all polynomials in the expressed way. It is easy to check that $I_{n,m}$ is an ideal of $\mathbf{R}\langle X \rangle$ and in the quotient algebra $A_{n,m} = \mathbf{R}\langle X \rangle / I_{n,m}$ we can identify all monomials not in $U_{n,m}$ with x_{n+1}. Indeed, the set of elements $v + I_{n,m}$, $v \in U_{n,m}$, and $x_{n+1} + I_{n,m}$ forms a basis of $A_{n,m}$ and is a multiplicatively closed set. Hence, $A_{n,m}$ is a finite dimensional stochastic algebra where Δ is the set of convex combinations of the indicated basic elements. Evidently, the set of ideals $I_{n,m}$, where $n, m = 1, \ldots, \infty$, is directed partially ordered, i.e., $I_{n,m} \supseteq I_{n',m'}$ for $n \leq n'$, $m \leq m'$, and their intersection is trivial, hence, $\mathbf{R}\langle X \rangle$ is an inverse limit of the family of finite dimensional stochastic algebras $A_{n,m}$ [7, p. 131]. ∎

In these notations, an *evolution law* of a class P is an equality $f = g$, where $f, g \in \Delta\langle X \rangle$, which holds for any values of variables in Δ_P.

For a family of population classes P we denote by $L(\mathsf{P})$ the set of all evolution laws which hold in any $P \in \mathsf{P}$. It is quite natural to describe it in some way for a given P.

Observe that $L(\mathsf{P})$ is closed under the following operations:

(i) Convex combination of two laws in $L(\mathsf{P})$,

$$\begin{aligned}&\textit{if } f = g,\ f' = g' \in L(\mathsf{P}), \textit{ and } 0 \leq \lambda \leq 1,\\ &\textit{then } \lambda f + (1-\lambda) f' = \lambda g + (1-\lambda) g' \in L(\mathsf{P});\end{aligned}$$

(ii) Multiplication by a polynomial in $\Delta\langle X \rangle$,

$$\textit{if } f = g \in L(\mathsf{P}) \textit{ and } h \in \Delta\langle X \rangle, \textit{ then } fh = gh \in L(\mathsf{P});$$

(iii) Replacing a variable with a polynomial in $\Delta\langle X \rangle$:

$$\begin{aligned}&\textit{if } f(x, \ldots) = g(x, \ldots) \in L(\mathsf{P}) \textit{ and } h \in \Delta\langle X \rangle,\\ &\textit{then } f(h, \ldots) = g(h, \ldots) \in L(\mathsf{P}).\end{aligned}$$

Starting with a number of laws in $L(\mathsf{P})$, one can produce other ones by means of these operations. Also, let $u \in V\langle X \rangle$. Then an evolution law $f = g$ holds in P if and only if a convex combination of $f = g$ and of the trivial law $u = u$ holds in P. Let $f' = g'$ be such law. It is natural to call $f = g$ and $f' = g'$ equivalent. Notice that for any P the set $L(\mathsf{P})$ is closed under this relation. Also, every equivalence class contains a single law $f = g$, where f and g are convex combinations of distinct monomials; such a law is called a *reduced* one.

A subset $S \subset L(\mathsf{P})$ is said to be *a set of generators*, if any law in $L(\mathsf{P})$ is equivalent to a law obtained from S by means of operations (i)–(iii) in a finite number of steps.

Certainly, generators of $L(\mathsf{P})$ for a population of a simple form may be found explicitly, but it should be rather difficult in a general case. In this work we study the problem of a *finite* set of generators of $L(\mathsf{P})$ for certain classes of populations.

Now we will show that this problem is reduced to that of identities of the corresponding stochastic algebra. A polynomial $f \in \mathbf{R}\langle X \rangle$ is said to be *an identity* of a stochastic algebra (A, Δ), if it vanishes under substitution of any elements in Δ. In other words, f belongs to the kernel of any morphism from $(\mathbf{R}\langle X \rangle, \Delta\langle X \rangle)$ to (A, Δ). We denote by $T(A, \Delta)$ the set of all identities of (A, Δ). If P is a family of population classes, then we denote by $T(\mathsf{P})$ the intersection of $T(A, \Delta_P)$, where $P \in \mathsf{P}$.

Notice that $T(\mathsf{P})$ is an ideal of $\mathbf{R}\langle X \rangle$ which is invariant under any morphism from $(\mathbf{R}\langle X \rangle, \Delta\langle X \rangle)$ to $(\mathbf{R}\langle X \rangle, \Delta\langle X \rangle)$ and its elements are identities of the one-dimensional algebra $\mathbf{R}$. We call such an ideal of $\mathbf{R}\langle \mathrm{X} \rangle$ *a stochastic T-ideal*. A subset S of a stochastic T-ideal I is a *set of generators*, if I is the minimal stochastic T-ideal which contains S. If such a set is minimal, then it is called a *basis* of the stochastic T-ideal I. A basis of $T(A, \Delta)$ is a basis of identities of (A, Δ). Notice that a basis of I always exists, if I has a finite set of generators.

PROPOSITION 2. *Let P be a family of population classes. Every polynomial in $T(\mathsf{P})$ has the form $\alpha(f - g)$ where $\alpha \in \mathbf{R}$ and $f = g$ is a reduced evolution law in $L(\mathsf{P})$. Also, in this way a set of generators of $T(\mathsf{P})$ corresponds to a set of generators of $L(\mathsf{P})$.*

Proof. Let $P \in \mathsf{P}$ and $\Delta = \Delta_P$. Take $h \in T(A, \Delta)$, then $h = \sum_u \alpha_u u$ where u runs over a finite subset of $V\langle X \rangle$, $\alpha_u \in F$. Since (A, Δ) is a *baric* algebra [18], i.e., there is a homomorphism ω: $A \to \mathbf{R}$, where $\omega(u) = 1$ for any $u \in \Delta$, the sum of positive coefficients $\alpha = \sum_{\alpha_u \geq 0} \alpha_u$ is equal to minus the sum of the negative ones. We put f/α (g/α), the sum (with minus) of those u which have α_u positive (negative).

Let S be a set of generators of $T(\mathsf{P})$. Then every identity of P is obtained from S by means of linear combination, multiplying by a monomial in $V\langle X \rangle$ and by replacing a variable with a polynomial in $\Delta\langle X \rangle$. These operations correspond to (i)–(iii). ∎

Notice that the quotient algebra $A = \mathbf{R}\langle X \rangle / \Gamma$, where Γ is a stochastic T-ideal may not be a stochastic algebra. However, it is still a baric algebra (A, ω). Also, notice that any stochastic T-ideal Γ is invariant under replacing a variable with a polynomial $g = \sum_{i=1}^n \alpha_i u_i$, where $u_i \in V\langle X \rangle$ and $\sum_i \alpha_i = 1$ (α_i is not necessarily positive). Indeed, fix a $f = f(x, \dots) \in \Gamma$

and consider $f_1 = f|_{x=g}$. Modulo Γ it has the form $f_1 = \sum_{j=1}^{m} \phi_j v_j$, where the ϕ_j are polynomials in $\alpha_1, \dots, \alpha_n$ and $v_j \in V\langle X \rangle$. Since $\phi_j = 0$ at any point of the open subset $\alpha_i > 0$ $(i = 1, \dots, n)$ of the hyperplane $\sum_i \alpha_i = 1$, this holds at any point of this hyperplane, which proves the claimed statement. In particular, $T(A, \Delta)$ is equal to the set $T(A, \omega)$ of all polynomials in $\mathbf{R}\langle X \rangle$ which vanish under replacing variables with elements in a hyperplane $\Pi(A) = \{a \in A | \omega(a) = 1\}$. For an arbitrary baric algebra (A, ω) the set $T(A, \omega)$ is also defined; obviously, it is a stochastic T-ideal and we call it the ideal of identities of a baric algebra (A, ω).

Conversely, any stochastic T-ideal Γ is the ideal of identities $T(A, \omega)$ of the baric algebra $(A/\Gamma, \omega)$. Therefore, we come to the problem of a finite basis of a *baric* T-ideal, i.e., an ideal of $\mathbf{R}\langle X \rangle$ which is closed under substitution of polynomials in $\Pi\langle X \rangle = \{\sum_i \alpha_i u_i \,|\, u_i \in V\langle X \rangle, \sum_i \alpha_i = 1\}$. It enables us to consider the case of an arbitrary field F. Although some statements are valid for algebras over an arbitrary field F, we will suppose that the characteristic of F is equal to zero.

Using the weight function ω, we may transform an identity of a baric algebra (A, ω) in another identity, involving ω, that holds for any values of variables in A. To do that, it is enough to replace each variable x by $x/\omega(x)$ and then multiply by the common denominator.

In this paper we consider the class of Bernstein populations. Recall that a baric algebra (A, ω) is a Bernstein one if it satisfies the identity

$$(x^2)^2 = x^2 \omega(x)^2. \tag{3}$$

The main results proved in this paper are

THEOREM 1. *Let (A, ω) be a Jordan–Bernstein or nuclear Bernstein algebra over a field of characteristic* 0. *Then the ideal $T(A, \omega)$ has a finite set of generators.*

THEOREM 2. *Let (A, ω) be a Jordan or nuclear Bernstein algebra. Then $T(A, \omega)$ is equal to the ideal of identities of the Grassmann envelope of a finitely generated baric superalgebra.*

Since the principal case is when A is an algebra generated by an infinite number of elements, we use methods of superalgebras developed in [11]. We extend the class of baric algebras in a natural way, then define a baric superalgebra and obtain a theorem that is the analogue to one proved in [11] for associative algebras.

In the final section we give examples of baric algebras which show that main conditions in the theorems are important.

2. IDENTITIES OF BERNSTEIN ALGEBRAS

Let (A, ω) be a Bernstein algebra. Then A has a nonzero idempotent e and decomposes with respect to e into a direct sum: $A = eF \dotplus U \dotplus Z$, where $U = \{a \in A \mid ae = 1/2a\}$ and $Z = \{a \in A \mid ae = 0\}$ (the Peirce decomposition). The following relations hold [18, p. 208],

$$U^2 \subseteq Z, \qquad UZ + Z^2 \subseteq U; \tag{4}$$

$$u^3 = u(uz) = uz^2 = (uz)(uz) = 0, \tag{5}$$

where $u \in U$, $z \in Z$.

A Bernstein algebra (A, ω) is called *nuclear* if $\ker(\omega)$ is generated by U. The subclass of Jordan–Bernstein algebras can be defined by the identity [17]

$$x^3 = x^2\omega(x). \tag{6}$$

We begin with the following evident statement.

LEMMA 1. *Let I be a finitely generated baric T-ideal. Any baric T-ideal $J \supseteq I$ has also a finite set of generators if and only if any ascending chain of baric T-ideals containing I become stationary.*

So, in order to prove Theorem 1, we need to find a *finitely generated* baric T-ideal which consists of identities valid in both classes and then to prove a.c.c. for baric T-ideals modulo this one.

LEMMA 2. *Let (B, ω) be a Jordan–Bernstein or nuclear Bernstein algebra. Then (B, ω) satisfies the identities*

$$(x^3 - \omega(x)x^2)y \cdot t - \frac{1}{2}(x^3 - \omega(x)x^2)t\omega(y) = 0 \tag{7}$$

$$(x^3 - \omega(x)x^2)(ty) - (x^3 - \omega(x)x^2)t\omega(y) = 0. \tag{8}$$

Proof. For a Jordan–Bernstein algebra, it follows from (6). If (B, ω) is a nuclear Bernstein algebra, then $x^3 - \omega(x)x^2 \in \operatorname{ann}(\ker \omega)$ (see [17]). So, if $y \in \ker \omega$ or $t \in \ker \omega$, the two identities are satisfied. Finally, if $t = y = e$, then they also hold. ∎

Denote by M the class of Bernstein algebras defined by (7, 8). The following statement will imply Theorem 1.

THEOREM 3. *Any (baric) algebra in M has a finite basis of identities over F.*

In [8, 3] it was proved that $\ker(\omega)^2$ is nilpotent for any Bernstein algebra. The following property of baric algebras in M is important for our proof.

PROPOSITION 3. *Let* $(B, \omega) \in \mathsf{M}$. *Then there is* $n_0 \in \mathbf{N}$ *such that any element in* $\ker(\omega)$ *generates a nilpotent ideal of index* $\le n_0$.

Proof. Let $B = Fe \dotplus U \dotplus Z$; it suffices to show that any product of elements in $\{e\} \cup U \cup Z$, where one element in $U \cup Z$ occurs at least n_0 times, is equal to zero. The occurrences of e can be replaced consequently with a scalar 1, 1/2, or 0. Hence, we need to bound the nilpotency index of the ideal of $\ker(\omega)$ generated by one element in it.

Notice that $N = \ker(\omega)$ satisfies the identity $x^3 y \cdot z = 0$ by (7). Denote $C = \{a \in N \mid aN \cdot N = 0\}$. Then C is an ideal of N and $\overline{N} = N/C$ is a commutative algebra satisfying the identity $x^3 = 0$ (*) Obviously, it suffices to show that any element in $\overline{N}$ generates a nilpotent ideal of a bounded index.

Notice that $\overline{N}^2$ is nilpotent [8, 3] of index $n \le 9$. Next, using the linearized identity (*) in the operator form $R_x R_y + R_y R_x = -R_{xy}$ (where R_x: $a \to ax$), one can easily check that the product of two (and consequently of any finite number) of ideals of $\overline{N}$ is an ideal. Indeed, let $\bar{I}_1, \bar{I}_2$ be two ideals of $\overline{N}$. Then

$$y_1 R_{y_2} R_x = -y_1 R_x R_{y_2} - y_1 R_{y_2 x} \in \bar{I}_1 \bar{I}_2$$

for all $y_1 \in \bar{I}_1, y_2 \in \bar{I}_2, x \in \overline{N}$. Moreover, if E is a product of several operators, where at least two of them are equal, then $(N^2)^k E \subseteq (N^2)^{k+1}$ (**).

Let us denote I the ideal of N generated by a, where $a \in N$. Let us prove that $I^{2^k} \subseteq (N^2)^k$. Obviously, $I^2 \subseteq N^2$. Next, for $k \ge 1$,

$$I^{2^{k+1}} = \sum_{i, j \ge 1,\, i+j = 2^{k+1}} I^i I^j \subseteq I^{2^{k+1}-1} I + \left(N^2\right)^k I^2$$
$$\subseteq \left(I^{2^k} I\right) I + \left(N^2\right)^{k+1} \subseteq \left(N^2\right)^{k+1},$$

where we have used induction on k and (**). Indeed, the term

$$I^{2^{k+1}-1} I \subset \left(I^{2^k} I\right) I \subset \left(N^{2^k} I\right) I \subset N^{2^{k+1}}$$

by (**), and all terms $I^i I^j$, where $i > 2^k$, $j \ge 2$ (or $i \ge 2$, $j > 2^k$) lie in $(N^2)^k N^2 = (N^2)^{k+1}$ by the induction assumption. Hence, $n_0 \le 2^9$. ∎

We will need some methods of superalgebras [11]. This forces us to extend the class of baric algebras.

3. GENERALIZED BARIC ALGEBRAS AND SUPERALGEBRAS

We may consider a baric algebra (A, ω) as an algebra with an additional multiplication $a * b = a\omega(b)$ which satisfies the identities

$$ab = ba; \tag{9}$$

$$a * (bc) = (a * b) * c; \tag{10}$$

$$(ab) * c = (a * c)b = a(b * c); \tag{11}$$

$$a * (b * c) = (a * b) * c. \tag{12}$$

On the other hand, if we have a commutative algebra A with an additional multiplication $a * b$ satisfying (9)–(12), then we may define a linear mapping from A to $\mathrm{End}_F A$, $a \to \omega(a)$, where $\omega(a)$ is the operator of the second multiplication by $a \in A$ from the right, $\omega(a)$: $x \to x * a$. Thanks to (10), it is a homomorphism; also, by (11), the image is an associative commutative subalgebra K of the centroid of the algebra A:

$$\Gamma(A) = \left\{\phi \in \mathrm{End}_F(A) \middle| \forall a, b \in A, (a\phi)b = a(b\phi) = (ab)\phi\right\}.$$

Moreover, by (12), ω is a homomorphism over K. In other words, A is a baric algebra over an associative-commutative algebra (with 1).

In order to introduce the notion of a baric superalgebra we consider the class of algebras equipped with an additional multiplication $a * b$. We call such an algebraic system A a *generalized* (g.) baric algebra and, sometimes, we will use the notation $(A, *)$.

Evidently, the class of such algebras is a variety of algebraic systems [7], hence, it has the free algebra $\mathsf{F}\langle X\rangle$ with the set of free generators X. The identical mapping from X to X can be extended to an epimorphism of g. baric algebras π: $\mathsf{F}\langle X\rangle \to (F\langle X\rangle, *) = (F\langle X\rangle, \omega)$.

PROPOSITION 4. *If Γ is a baric T-ideal in $F\langle X\rangle$, then the maximal homogeneous subspace I in $\pi^{-1}(\Gamma)$ is a T-ideal of $\mathsf{F}\langle X\rangle$. Also, if $\Gamma \subseteq \Gamma'$, $\Gamma \neq \Gamma'$, then $I \subseteq I'$, $I \neq I'$.*

Proof. Evidently, I is closed under multiplication by any element in $\mathsf{F}\langle X\rangle$, i.e., if $f \in I$ and $a \in \mathsf{F}\langle X\rangle$, then $fa, af, f * a, a * f \in I$. Next, let us show that I is closed under replacing a variable with any element $h \in \mathsf{F}\langle X\rangle$.

Take a homogeneous $f \in I$, where the degree in a variable x, $\deg_x(f)$, is equal to m. Then for any $h \in \mathsf{F}\langle X\rangle$, where $\omega(\pi(h)) = 1$, and $\alpha \in F$ we have

$$f|_{x=\alpha h} = \alpha^m f|_{x=h} \in \pi^{-1}(\Gamma),$$

where $f|_{x=h}$ denotes the polynomial obtained from h when x is substituted by h. In particular, for any $x, y \in X$ and $\alpha \in F$, we have $f|_{x=x+\alpha y} \in \pi^{-1}(\Gamma)$. Hence (cf. [20, p. 9]) any homogeneous component of $f_1 = f|_{x=x+y}$ lies in $\pi^{-1}(\Gamma)$, hence, $f_1 \in I$. Obviously, this also holds for any $f \in I$. Using it several times, we get $f|_{x=x_1+\cdots+x_n} \in I$ for any $x_1, \dots, x_n \in X$. Since any $h \in \mathsf{F}\langle X\rangle$ has the form $h = \sum_{j=1}^{n} \alpha_j h_j$, where $\pi(h_j) \in V\langle X\rangle$ (π is an epimorphism), we get $f|_{x=h} \in I$.

To prove the second statement it suffices to show that $\pi(I) = \Gamma$. Take $f = f(x_1, \dots, x_n) \in \Gamma$ and put $k_i = \deg_{x_i} f$. The f is decomposed into the sum of its homogeneous components $g = g_{m_1, \dots, m_n}$, where $m_i = \deg_{x_i} g$. Letting

$$f' = \sum g_{m_1,\dots,m_n} \omega(x_1)^{k_1 - m_1} \dots \omega(x_n)^{k_n - m_n},$$

we get $\pi(f') = f$ and f' is homogeneous, hence, $f' \in I$. ∎

Thus, we have reduced our problem to a.c.c. of T-ideals of $\mathsf{F}\langle X\rangle$ modulo the ideal of identities of algebras in M (in other words, we need to prove that the variety M has Specht's property [11]).

Let P_n be the set of all multilinear polynomials in $\mathsf{F}\langle X\rangle$ of degree n. The symmetric group S_n acts on P_n

$$\sigma \cdot g(x_1, \dots, x_n) = g(x_{\sigma(1)}, \dots, x_{\sigma(n)}), \qquad \sigma \in S_n,\ g(x_1, \dots, x_n) \in P_n \tag{13}$$

so, P_n is a module over the group algebra $F[S_n]$. In order to study identities of algebras, we will need Young diagrams (see [6]).

Let n be a natural number and let $\tilde{n} = (n_1, \dots, n_k)$ be a partition of n, that is, a sequence of natural numbers such that $n_1 \geq n_2 \geq \cdots \geq n_k$ and $\sum_{i=1}^{k} n_i = n$. A Young tableau D associated to $\tilde{n}$ is a figure such that the ith row has n_i cells. If we put in the cells of D different numbers from $1, \dots, n$ in some way, then we get a Young diagram D^*. Let $P_{D^*}(Q_{D^*})$ be the subgroup of S_n generated by all permutations which leave invariant the rows (columns) of D^*. Then the element

$$e_{D^*} = \sum_{p \in P_{D^*}} \sum_{q \in Q_{D^*}} \operatorname{sgn}(q) pq, \tag{14}$$

where $\operatorname{sgn}(q)$ is the sign of a permutation q, generates a minimal right ideal of the group algebra $F[S_n]$. Moreover, every irreducible S_n-module is isomorphic to such a one and the S_n-module P_n is the direct sum of such modules.

Let Λ be a subset of $\{x_1, \ldots, x_n\}$. Then the symmetric group $S(\Lambda)$ is embedded in $S_n = S(\{x_1, \ldots, x_n\})$: if $x_i \notin \Lambda$, then any permutation in $S(\Lambda)$ acts trivially on x_i. Denote $S_\Lambda^+ = \sum_{\sigma \in S(\Lambda)} \sigma$, $S_\Lambda^- = \sum_{\sigma \in S(\Lambda)} \operatorname{sgn}(\sigma)\sigma$. Then

$$e_{D^*} = S_{\Lambda_1}^+ \cdots S_{\Lambda_k}^+ S_{\Lambda'_1}^- \cdots S_{\Lambda'_l}^-, \tag{15}$$

where Λ_i (Λ'_i) is the set of variables in the ith row (column). Also, let $D_{t,s}$ be the Young tableau which has t rows each of them of length s.

PROPOSITION 5. *Let (A, ω) be a Bernstein algebra which is considered the g. baric algebra $(A, *)$. If there exists a subspace $V \subseteq A$ of codimension m such that for all $v \in V$, $\mathrm{idl}(v)$ is nilpotent of index $\leq n_0$, then for any Young diagram D^* with $D_{t+1,s+1} \subseteq D$, $t \geq m$, and $s \geq (t+1)n_0/(t+1-m) - 1$, we have $e_{D^*} \cdot P_n \subset T(A, *)$.*

Proof. Let $f = e_{D^*} g$, where $g \in P_n$. We will prove that for any $\bar{x}_1, \ldots, \bar{x}_n \in A$, $f(\bar{x}_1, \ldots, \bar{x}_n) = 0$. Suppose this is not the case for some $\bar{x}_1, \ldots, \bar{x}_n$. We have $A = V \dotplus W$, where W is a subspace of dimension m spanned by $w_1, \ldots, w_m$. The element f is multilinear, hence, we may assume that $\bar{x}_i \in V \cup \{w_1 \ldots, w_m\}$.

We use notations of (15). If a row Λ_j has more than n_0 cells with elements in V, then $\bar{f} = f(\bar{x}_1, \ldots, \bar{x}_n) = 0$, because f is a linearization of a polynomial $h(y, \ldots) = f|_{x=y\ \forall x \in \tilde{\Lambda}_j}$, where $\tilde{\Lambda}_j$ is the subset of variables of Λ_j with values in V, and $h(\bar{y}, \ldots) \in \mathrm{idl}(\bar{y})^{n_0} = 0$. Therefore, the number of cells where $\bar{x}_i \in V$ in the first $t+1$ rows is less than $(t+1)n_0$.

On the other hand, there is $\sigma \in S(\Lambda_1) \ldots S(\Lambda_k)$ such that $g_1(\bar{x}_1, \ldots, \bar{x}_n) \neq 0$, where

$$g_1 = \sigma S_{\Lambda'_1} \ldots S_{\Lambda'_l} g = S_{\sigma(\Lambda'_1)} \ldots S_{\sigma(\Lambda'_l)} \sigma g.$$

The number of variables $x_i \in \sigma(\Lambda'_j)$ such that $\bar{x}_i \in W$ does not exceed m. The length of the first $s+1$ columns is greater than t, therefore, for each j, $1 \leq j \leq s+1$, in $\sigma(\Lambda'_j)$ there are at least $(s+1)(t+1-m)$ cells with i such that $\bar{x}_i \in V$.

Since σ keeps the rows invariant, $(s+1)(t+1-m) < (t+1)n_0$, which contradicts the hypothesis. ∎

By Proposition 3 for any algebra in M we may set $m = 1$, hence, $t = 1$ and $s = 2n_0 - 1$.

Representations of the symmetric group are important tools in the theory of algebras with identities (see, for example, [15]). In our proof we will use the method of superalgebras developed mostly by A. R. Kemer [10, 11] (also, see [2]). He showed that a proper subvariety of associative algebras is defined by the Grassmann envelope of an appropriate *finitely generated* associative superalgebra. In this way, one of the key technical results is the following statement.

THEOREM 4. *Let A be an algebra over F. Suppose that there exist $t, s \in \mathbf{N}$ such that for any Young diagram D^*, $D \supseteq D_{t+1,s+1}$, and multilinear polynomial f we have $e_{D^*} \cdot f \in T(A)$. Then the ideal of identities of A is equal to that of the Grassmann envelope of a $\mathbf{Z}_2$-graded algebra generated by t even and s odd elements.*

In other words, the variety of algebras generated by A is of superrank (t,s) (a definition suggested by I. P. Shestakov).

Although A. R. Kemer considered the associative case, this statement is valid for an arbitrary variety of algebras. Moreover, the result can be extended to the class of algebras where we have several multilinear operations instead of one multiplication and now let us prove it.

First, observe that in an algebra with several multilinear operations over a field of characteristic zero, every T-ideal is generated by its multilinear polynomials. As for g. baric algebras, we denote by P_n the set of multilinear polynomials of degree n and the symmetric group S_n acts on P_n by (13).

A $\mathbf{Z}_2$-graded algebra is an algebra $A = A_0 \dotplus A_1$ such that for every multilinear operator p_i

$$p_i\left(A_{i_1}, \ldots, A_{i_r}\right) \subseteq A_{i_1 + \cdots + i_r},$$

where $i_1, \ldots, i_r \in \mathbf{Z}_2$. An element from A_0 (A_1) is said to be an even (odd) element. Such a grading can be defined by means of an automorphism ψ of order 2: $A_i = \{a \in A \mid \psi(a) = (-1)^i a\}$, $i \in \mathbf{Z}_2$.

Let $F\langle X \rangle$ be the free algebra with the set of free generators $X = Y \cup Z$, where Y, Z are disjoint infinite sets of variables. We define an automorphism ψ on $F\langle X \rangle$ such that $\psi(y) = y$ and $\psi(z) = -z$; so $F\langle X \rangle$ becomes a $\mathbf{Z}_2$-graded algebra.

Let G be the Grassmann algebra, i.e., an associative algebra generated by 1, $e_1, \ldots, e_n, \ldots,$ where $e_i e_j = -e_j e_i$ for any i, j. It has a $\mathbf{Z}_2$-grading $G = G_0 \dotplus G_1$, where G_0 is the vector space generated by 1 and all products of an even number of generators e_i, and G_1 is spanned by odd products of elements e_i.

A graded ideal $I \subseteq F\langle X \rangle$ is called a T_2-ideal if for any $f(y_1, \ldots, y_t; z_1, \ldots, z_s) \in F\langle X \rangle$, $u_i \in F\langle X \rangle_0$ and $v_j \in F\langle X \rangle_1$, the element $f(u_1, \ldots, u_t; v_1, \ldots, v_s) \in I$. Also, a graded ideal I is an S_2-ideal if for any $f(y_1, \ldots, y_t; z_1, \ldots, z_s) \in F\langle X \rangle$, u_i is in the linear span of Y and v_j in the linear span of Z, the element $f(u_1, \ldots, u_t; v_1, \ldots, v_s) \in I$.

It is clear that any T_2-ideal is the ideal of graded identities of a $\mathbf{Z}_2$-graded algebra. In turn, an S_2-ideal I can be considered as the ideal of identities for elements of a graded subspace V of a $\mathbf{Z}_2$-graded algebra A, for example, $A = F\langle X\rangle/I$, $V = \mathrm{vect}_F X/I$; in other words, I is the ideal of identities of a $\mathbf{Z}_2$-graded pair (A, V). Notice that any S_2-ideal is homogeneous and generated by multilinear polynomials (cf. [20, Sect. 1]). Also, the class of such pairs is a variety of algebraic systems.

Let $B = Y$ or $B = Z$. Denote by $I^{+}_{B,r}$ an S_2-ideal generated by all elements of the form

$$S^{+}_{\Lambda} f \tag{16}$$

and by $I^{-}_{B,r}$ an S_2-ideal generated by

$$S^{-}_{\Lambda} f, \tag{17}$$

where $f = f(b_1, \dots, b_r, \dots)$ is a multilinear polynomial in $F\langle X\rangle$ and $\Lambda = \{b_1, \dots, b_r\} \subseteq B$.

Let Γ_1, Γ_2 be ideals of $F\langle X\rangle$. We say Γ_1 is T-equivalent to Γ_2, and it is denoted by $\Gamma_1 \sim_T \Gamma_2$, if the biggest T-ideals contained in Γ_1 and Γ_2 are equal.

PROPOSITION 6 (cf. [11, Proposition 1.3, Case 2]). *Let Γ be a T-ideal and $t, s \in \mathbf{N}$. Suppose that for any Young diagram D^* such that $D_{t+1,s+1} \subseteq D$ we have $e_{D^*}P_n \subseteq \Gamma$. Then*

$$\Gamma \sim_T \Gamma + I^{-}_{Y,t+1} + I^{+}_{Z,s+1}.$$

Proof. Denote by Γ' the maximal T-ideal contained in the right hand side. Obviously, $\Gamma \subseteq \Gamma'$ and we need to prove the reciprocal inclusion.

Take a multilinear $f \in \Gamma' \cap P_n$. Since the unit $1 \in F[S_n]$ is a linear combination of elements of the form (14), it suffices to show that for any Young diagram D^* the element $e_{D^*}f \in \Gamma$. By the hypothesis, this holds for $D \supseteq D_{t+1,s+1}$. Now, suppose that it does not hold for some $D \not\supseteq D_{t+1,s+1}$. Then there are $t', s', t' \le t, s' \le s$, such that the union of the first s' columns and the first t' rows contains D.

Observe that if Λ, Λ' are disjoint subsets of $\{1, \dots, n\}$, then any element in $S(\Lambda)$ commutes with $S(\Lambda')$. We use further notations of (15).

Denote by Ω the set of numbers in the first t' rows, i.e., $\Omega = \bigcup_{i=1}^{t'} \Lambda_i$. Also, let $\Lambda''_i = \Lambda'_i \setminus \Omega$ be the set of all numbers in the ith column which do not belong to the set of numbers in the first t' rows. Then $S^{-}_{\Lambda'_i} = \sum_{\mu} \mathrm{sgn}(\mu) S^{-}_{\Lambda''_i} \mu$, where μ runs over the set of representatives of right classes in $S(\Lambda'_i)$ with respect to the subgroup $S(\Lambda''_i)$. Since μ commutes with $S(\Lambda'_j)$, $j \neq i$, we have $e_{D^*} = \sum_{\tau,\sigma} \alpha_{\tau,\sigma} \tau S \sigma$ where

$$S = S^+_{\Lambda_1} \cdots S^+_{\Lambda_{t'}} S^-_{\Lambda''_1} \cdots S^-_{\Lambda''_{s'}}$$

and τ, σ run over a subset of S_n, $\alpha_{\tau,\sigma} \in F$.

There are $\tau, \sigma \in S_n$ such that $\tau S \sigma f \in \Gamma' \setminus \Gamma$. Denote $f_1 = \sigma f$; obviously, $f_1 \in \Gamma' \setminus \Gamma$. Also, let $\Omega'' = \bigcup_{i=1}^{s'} \Lambda''_i$. Since Γ', Γ are T-ideals and $\Omega \cap \Omega'' = \emptyset$, we may assume that $\Omega \subseteq Y$ and $\Omega'' \subseteq Z$. Also, $f_1 = g + h_1 + h_2$, where $g \in \Gamma$, $h_1 \in I^-_{Y,t+1}$, and $h_2 \in I^+_{Z,s+1}$. It suffices to show that $Sh_i = 0$.

Consider the case $i = 1$ (for $i = 2$ it is similar). We may assume that h_1 is a generator of the form (17), i.e., $h_1 = S^-_{\Lambda} g_1$ for some multilinear polynomial g_1 depending on variables in $\Lambda \subseteq Y$, $|\Lambda| = t + 1$. Then

$$\begin{aligned} S \cdot S^-_{\Lambda} &= S^+_{\Lambda_1} \cdots S^+_{\Lambda_{t'}} S^-_{\Lambda''_1} \cdots S^-_{\Lambda''_{s'}} S^-_{\Lambda} \\ &= S^-_{\Lambda''_1} \cdots S^-_{\Lambda''_{s'}} S^+_{\Lambda_1} \cdots S^+_{\Lambda_{t'}} S^-_{\Lambda}. \end{aligned}$$

By the choice of s', t', we have $\Omega \cup \Omega'' = \{1, \dots, n\}$ and, hence, $\Lambda \subset \Omega$. Recall that $|\Lambda| = t + 1$; hence, there is j such that Λ_j contains at least two numbers in Λ, therefore, $S^+_{\Lambda_j} S^-_{\Lambda} = 0$ and $S \cdot S^-_{\Lambda} = 0$. ∎

Now, we are going to define a mapping $I \to I^*$ on the set of S_2-ideals of $F\langle X \rangle$. Let I be a S_2-ideal, then, as we noted above, it is the ideal of identities $T_2(A, V)$ of a $\mathbf{Z}_2$-graded pair (A, V).

Consider the tensor product of spaces $A \otimes_F G$ and define the operations on it letting

$$p_i(a_1 \otimes g_1, \dots, a_r \otimes g_r) = p_i(a_1, \dots, a_r) \otimes_F g_1 \cdots g_r.$$

The *Grassmann envelope* $G(A)$ of a $\mathbf{Z}_2$-graded algebra A is a $\mathbf{Z}_2$-graded subalgebra $A_0 \otimes G_0 + A_1 \otimes G_1$ of the algebra $A \otimes_F G$. It contains a $\mathbf{Z}_2$-graded subspace $G(V) = V_0 \otimes G_0 + V_1 \otimes G_1$. We define I^* as the set of graded identities of $(G(A), G(V))$.

Let us show that the definition is correct, i.e., it does not depend on the choice of the pair (A, V).

LEMMA 3. *Let $(A, V), (A', V')$ be $\mathbf{Z}_2$-graded pairs such that $T_2(A, V) = T_2(A', V')$. Then $T_2(G(A), G(V)) = T_2(G(A), G(V))$.*

Proof. It suffices to prove that $T_2(G(A), G(V)) \subseteq T_2(G(A'), G(V'))$ for any (A', V') in the variety of pairs defined by (A, V). By Birkhoff's theorem [7], any such a pair is obtained from (A, V) by means of standard algebraic constructions (direct products, subpairs, quotient pairs) in a finite number of steps.

Notice that the Grassmann envelope of the direct product of a given family of pairs is embedded into that of the Grassmann envelopes of the family. Similarly, the Grassmann envelope of a subpair lies in that of the pair. Finally, the Grassmann envelope of a quotient pair $(A'/I', V/I')$, where I' is a graded ideal of A', is isomorphic to $(G(A')/G(I'), G(V')/G(I'))$. Hence, on each step we get the necessary inclusion. ∎

LEMMA 4. *Let I, J be S_2-ideals. Then*

(1) $(I^*)^* = I$,

(2) *if* $I \subseteq J$, *then* $I^* \subseteq J^*$,

(3) $(I + J)^* = I^* + J^*$,

(4) $(I^-_{Y,m})^* = I^-_{Y,m}, (I^+_{Z,n})^* = I^-_{Z,n}, (I^-_{Z,n})^* = I^+_{Z,n}$.

Proof. Let V be a $\mathbf{Z}_2$-graded vector subspace $\mathrm{vect}\langle x/I \,|\, x \in X\rangle \subset F\langle X\rangle/I$ of $A = F\langle X\rangle/I$. Then $T_2(A, V) = I$. Since $G(G(A)) = A_0 \otimes G_0 \otimes G_0 + A_1 \otimes G_1 \otimes G_1 \subseteq A \otimes G(G)$ and $G(G)$ is a commutative algebra, we have $I \subseteq (I^*)^*$. Conversely, let $f(y_1, \dots, y_m, z_1, \dots, z_n)$ be a multilinear polynomial in $(I^*)^*$. Then it is a graded identity of $(G(G(A)), G(G(V)))$. Hence, if $e_1, \dots, e_n \in G_1$ are generators of G, then

$$\begin{aligned} 0 &= f(y_1 \otimes 1 \otimes 1, \dots, y_m \otimes 1 \otimes 1, z_1 \otimes e_1 \otimes e_1, \dots, z_n \otimes e_n \otimes e_n) \\ &= f(y_1, \dots, y_m, z_1, \dots, z_n) \otimes ((e_1 \otimes e_1) \cdots (e_n \otimes e_n)) \\ &= f(y_1, \dots, y_m, z_1, \dots, z_n) \otimes e_1 \cdots e_n \otimes e_1 \cdots e_n = 0 \end{aligned}$$

for arbitrary $y_i \in V_0, z_j \in V_1$, hence, $f \in I$.

The second part easily follows from the definition of the mapping $I \to I^*$.

Next, by (2), we have $I^* + J^* \subseteq (I + J)^*$ and thanks to (1), $I + J = (I^*)^* + (J^*)^* \subseteq (I^* + J^*)^*$. Applying $*$ to the latter relation, we get $(I + J)^* \subseteq I^* + J^*$.

Now, let $f = f(y_1, \dots, y_m, z_1, \dots, z_n)$ be a multilinear graded polynomial which generates an S_2-ideal I. Then I^* is generated by f^* which can be found in the following way. Take $h_1, \dots, h_m \in G_0, g_1, \dots, g_n \in G_1$ such that their product is not zero. Then for any multilinear $f_1(y_1, \dots, y_m, z_1, \dots, z_n)$

$$f_1(y_1 \otimes h_1, \dots, y_m \otimes h_m, z_1 \otimes g_1, \dots, z_n \otimes g_n) = f_1^* \otimes h_1 \cdots h_m g_1 \cdots g_n \tag{18}$$

for a certain multilinear f_1^*, which does not depend on the choice of h_i, g_j. If $f_1^* \in I$, then $f_1 \in I^*$. Also, $(f_1^*)^* = f_1$, hence, by (1)–(2), f^* generates I^*.

The final part follows from (18) and (1)–(3). Actually, if $f = f(y_1, \ldots, y_m, z_1, \ldots, z_n)$ is a multilinear polynomial and $\Lambda \subseteq \{y_1, \ldots, y_m\}$ ($\Lambda \subseteq \{z_1, \ldots, z_n\}$), then $(S_\Lambda^- f)^* = S_\Lambda^- f^*$ ($(S_\Lambda^- f)^* = S_\Lambda^+ f^*$ and $(S_\Lambda^+ f)^* = S_\Lambda^- f^*$, respectively. ∎

PROPOSITION 7 (cf. [11, Proposition 1.2]). *Let I be an S_2-ideal of $F\langle X\rangle$ such that $I_{Y,t+1}^- + I_{Z,s+1}^- \subseteq I$. Let Γ be the maximal T_2-ideal in I. Then it is the ideal of identities of a $\mathbf{Z}_2$-graded algebra B generated by t even and s odd elements.*

Proof. Let R be the free $\mathbf{Z}_2$-graded algebra generated by t even and s odd variables and put $B = R/\Gamma \cap R$. Obviously, $\Gamma \subseteq T_2(B)$. Conversely, suppose $T_2(B) \not\subseteq \Gamma$. Then $T_2(B) \not\subseteq I$ and, since $T_2(B)$, I are S_2-ideals, there is a multilinear polynomial $f = f(y_1, \ldots, y_n, z_1, \ldots, z_m)$ in $T_2(B) \setminus I$. Since S_n acts on even variables and for any Young diagram D^* which has a column of length $> t$ we have $e_{D^*} f \in I_{Y,t+1}^-$, we may assume that the set of variables $y_1, \ldots, y_n$ is broken down into $\leq t$ disjoint subsets and f is symmetric with respect to each one. Up to enumeration they are

$$y_1, \ldots, y_{n_1}; y_{n_1+1}, \ldots, y_{n_1+n_2}; \ldots; y_{n_1+\cdots+n_{t-1}+1}, \ldots, y_n.$$

Hence, f is the complete linearization of the element

$$f_1 = f\Big(\underbrace{y_1, \ldots, y_1}_{n_1}, \ldots, \underbrace{y_t, \ldots, y_t}_{n_t}, z_1, \ldots, z_m\Big) \tag{19}$$

in $T_2(B)$. The same holds for the odd elements and in (19) we can replace them with $z_1, \ldots, z_s$. Since $f_1 \in T_2(B)$, we have $f_1|_{\forall i,j\ y_i = \bar{y}_i,\, z_j = \bar{z}_j} = 0$, where $\bar{y}_i = y_i/\Gamma \cap R$ and $\bar{z}_j = z_j/\Gamma \cap R$, i.e., $f_1 \in \Gamma$ and, therefore, $f \in \Gamma$. ∎

Proof of Theorem 4. By Proposition 6,

$$\Gamma + I_{Y,t+1}^- + I_{Z,s+1}^+ \sim_T \Gamma. \tag{20}$$

Then, using Lemma 4, we get $I = (\Gamma + I_{Y,t+1}^- + I_{Z,s+1}^+)^* = \Gamma^* + I_{Y,t+1}^- + I_{Z,s+1}^-$. Next, let Γ_1 be the biggest T_2-ideal contained in I. By Proposition 7, Γ_1 is the ideal of graded identities of a (t, s)-generated $\mathbf{Z}_2$-graded algebra B. Moreover, Γ^* is a T_2-ideal, hence, $\Gamma^* \subseteq \Gamma_1 \subseteq I$. Applying *, we get

$$\Gamma \subseteq \Gamma_1^* \subseteq I^* = \Gamma + I_{Y,t+1}^- + I_{Z,s+1}^+.$$

Since $\Gamma \sim_T I^*$, the Γ is the maximal T-ideal contained in Γ_1^*. Notice that $T_2(B)$ is the ideal of graded identities of the pair (B, B). Hence, by Lemma 3, $\Gamma_1^* = T_2(G(B), G(B)) = T_2(G(B))$, hence, $T(G(B)) = \Gamma$. ∎

4. SPECHT'S PROPERTY IN A CLASS OF BERNSTEIN ALGEBRAS

In this section we will prove Theorem 3.

Let $\mathcal{M}$ be a variety of g. baric algebras. A $\mathbf{Z}_2$-graded g. baric algebra A is called an $\mathcal{M}$-superalgebra, if its Grassmann envelope $G(A)$ belongs to $\mathcal{M}$ (cf. [16, 14]). If $\mathcal{M}$ is defined by multilinear identities $\{f_i \mid i \in \Omega\}$, then the variety of $\mathcal{M}$-superalgebras is defined by identities obtained in the following way.

First, we fix a decomposition of the set of free variables into the infinite subsets of odd and even variables, $X = Y \cup Z$. Then we take an $f_i = f_i(x_1, \dots, x_n)$ and replace a number of its variables with even variables and the others with odd ones. We get a graded polynomial which is an identity of any $\mathbf{Z}_2$-graded algebra in $\mathcal{M}$. Notice that in this way we get a set of generators of the ideal I of identities $\mathcal{M}$ considered as a T_2-ideal.

Now, we claim that I^* is the ideal J of graded identities of the variety of $\mathcal{M}$-superalgebras. Actually, by definition, $I \subseteq J^*$, hence, by Lemma 4, $I^* \subseteq J$. On the other hand, $(I^*)^* = I$, hence, I^* is the ideal of graded identities of an $\mathcal{M}$-superalgebra, hence, $J \subseteq I^*$.

For example, the variety of commutative superalgebras is defined by the following graded identities

$$y_1 y_2 = y_2 y_1; \qquad y_1 z_2 = z_2 y_1; \qquad z_1 z_2 = -z_2 z_1.$$

Using the notation $\bar{x} = i \in \mathbf{Z}_2$ for the parity of an element x, we may write it in a shorter way:

$$ab = (-1)^{\bar{b}\bar{a}} ba. \tag{21}$$

From (9)–(12), by means of (18), we obtain graded identities which define together with (21) the variety of baric superalgebras:

$$a * (bc) = (a * b) * c; \tag{22}$$

$$a * (b * c) = (a * b) * c; \tag{23}$$

$$(ab) * c = (-1)^{\bar{b}\bar{c}} (a * c) b = a(b * c). \tag{24}$$

It means that A is a commutative superalgebra and the mapping $a \to \omega(a)$, where $\omega(a)$: $b \to b * a$ is a graded homomorphism from A to $\mathrm{End}(A) = \mathrm{End}_0(A) + \mathrm{End}_1(A)$, where

$$\mathrm{End}_i(A) = \left\{ \phi \in \mathrm{End}(A) \,\middle|\, A_\epsilon \phi \subset A_{i+\epsilon} \right\}.$$

Moreover, the image lies in the super centroid of A,

$$\left\{\phi = \phi_0 + \phi_1 \middle| \forall a, b \in A_0 \cup A_1, (ab)\phi_i = a(b\phi_i) = (-1)^{\bar{b}i}(a\phi_i)b\right\},$$

and $\omega(a\omega(b)) = \omega(a)\omega(b)$ for any $a, b \in A$.

Conversely, any commutative superalgebra with such a homomorphism can be considered as a baric superalgebra. For example, take a tensor space product $B\langle Y, Z\rangle$ of the free commutative superalgebra $F\langle Y, Z\rangle$ and the free associative-commutative superalgebra $F[Y, Z]$ with 1 and define multiplications on this vector space,

$$a \otimes b \cdot c \otimes d = (-1)^{\bar{b}\bar{c}} ac \otimes bd, \qquad a \otimes b * c \otimes d = a \otimes b\pi(c)d,$$

where π is the canonical homomorphism from $F\langle Y, Z\rangle$ onto $F[Y, Z]$, $\pi|_{Y \cup Z} = \mathrm{id}$.

Letting

$$B\langle Y, Z\rangle_0 = F\langle Y, Z\rangle_0 \otimes F[Y, Z]_0 + F\langle Y, Z\rangle_1 \otimes F[Y, Z]_1,$$
$$B\langle Y, Z\rangle_1 = F\langle Y, Z\rangle_0 \otimes F[Y, Z]_1 + F\langle Y, Z\rangle_1 \otimes F[Y, Z]_0,$$

we get a g. baric superalgebra $(B\langle Y, Z\rangle, *)$, satisfying (21)–(22), which is obviously a free one with the graded set of free generators $X = Y \cup Z$. We shall use the notations R_a: $x \to ax$ and $\omega(x)$: $x \to x * a$ for the operators of the right multiplication by an element a in a (super) baric algebra.

In the variety of baric algebras defined by Bernstein algebras (7), (8) (we use the same notation M) the following identities hold (this is the linearized form of (7)–(8)),

$$J(x, y, z)a \cdot b = 1/2J(x, y, z)\omega(a) \cdot b + Q(x, y, z)(a - 1/2\omega(a)) \cdot b; \tag{25}$$

$$J(x, y, z) \cdot ab = 1/2J(x, y, z)\omega(a) \cdot b + Q(x, y, z) \cdot (a - 1/2\omega(a))b, \tag{26}$$

where,

$$J(x, y, z) = (xy)z + (zx)y + (yz)x,$$
$$Q(x, y, z) = (xy)\omega(z) + (zx)\omega(y) + (yz)\omega(x).$$

Therefore, by (18) an M-superalgebra (i.e., a $\mathbf{Z}_2$-graded algebra the Grassmann envelope of which lies in M) satisfies the identities (25)–(26)

where $J(x, y, z)$ ($Q(x, y, z)$) is replaced with

$$J^*(x, y, z) = (xy)z + (-1)^{\bar{z}(\bar{x}+\bar{y})}(zx)y + (-1)^{\bar{x}(\bar{y}+\bar{z})}(yz)x$$

(respectively, $Q^*(x, y, z)$), i.e., we get the super analog of (25)–(26).

By Theorem 4 and Propositions 3 and 5, there are $t, s \in \mathbf{N}$ such that any subvariety N of M is defined by the Grassmann envelope of an M-superalgebra $A(\mathsf{N})$ of superrank (t, s).

We may assume that $A = A(\mathsf{N})$ is the quotient superalgebra $B\langle Y_t, Z_s\rangle/T$, where T is the ideal of graded identities of A in variables $Y_t \cup Z_s$, where $|Y_t| = t$, $|X_s| = s$. Hence, a.c.c. for T_2-ideals of $B\langle Y_t, Z_s\rangle$ modulo M-super identities will complete the proof.

Denote $B = B\langle Y_t, Z_s\rangle$, the subalgebra of $B\langle Y, Z\rangle$ generated by $Y_t \cup Z_s$. Also, let $B_k = F\langle Y_t, Z_s\rangle \otimes F_k[Y_t, Z_s]$, where $F_k[Y_t, Z_s]$ is the subspace of polynomials of degree $\leq k$. For a T_2-ideal I in B and for any $k \geq 0$ put

$$m_k(I) = \left\{ f \in B_k \,\middle|\, \exists f_i \in B_i,\ i = k+1, \dots, m;\ f + \sum_{i=k+1}^{m} f_i \in I \right\}.$$

Let $f \in B\langle Y, Z\rangle$ and $x, y \in Y$ ($x, y \in Z$), denote by $f\Delta_x(y)$ the homogeneous component of $f|_{x=x+y}$ of degree 1 in y (cf. [20, p. 14]) and $f\Delta_x(u) = f\Delta_x(y)|_{y=u}$, where $u \in B_0 \cup B_1$ and its parity is equal to that of x. It is easy to check that $\Delta_x(u)$: $f \to f\Delta_x(u)$ is a derivation of $B\langle Y, Z\rangle$, in particular, $[R_v, \Delta_x(u)] = R_{v\Delta_x(u)}, [\omega(v), \Delta_x(u)] = \omega(v\Delta_x(u))$.

The following properties follow easily from the definition of $m_k(I)$.

Lemma 5. *The set $m_k(I)$ is a vector subspace with the following properties*:

(1) *If $u \in F\langle Y_t, Z_s\rangle$, then $m_k(I)u \subseteq m_k(I)$*;

(2) *If $u \in F\langle Y_t, Z_s\rangle$ and $\omega(u) = 0$, then $m_k(I)\Delta_x(u) \subseteq m_k(I)$*;

(3) *If $x \in Y_t \cup Z_s$ then $m_k(I)\omega(x) \subseteq m_{k+1}(I)$.*

Define $m(I) = \sum_{k=0}^{\infty} m_k(I)$ for a T_2-ideal I of B.

Lemma 6. *Let I, J be T_2-ideals of $B\langle Y_t, Z_s\rangle$ such that $I \subseteq J$ and $m(I) = m(J)$. Then $I = J$.*

Proof. On the contrary, take a homogeneous $f = \sum_{i=k}^{m} f_i \in J \setminus I$ of degree n, where $f_i \in B_i$. Since $m < n$ (where $f_m \neq 0$), we may assume that

k is maximal for a fixed n. There is $g = \sum_{i=k}^{m'} g_i \in I$ such that $g_k = f_k$. Since I is homogeneous, we also may assume that $\deg(g) = n$. Hence, $h = f - g \in J \setminus I$ and $h = \sum_{i=k+1}^{n} h_i$, this contradicts the choice of f. ∎

Proof of Theorem 3. Let I_i, $i \in \mathbf{N}$ be an ascending chain of T_2-ideals of B which contain the ideal I_0 of M-super identities. It suffices to show that the chain $m(I_i)$, $i \in \mathbf{N}$, becomes stationary.

Let $J_1 = F\langle Y_t, Z_s\rangle^2$ and let J_k, $k \geq 2$, be the ideal of $F\langle Y_t, Z_s\rangle$ generated by $\sum_{i,j\geq 1, i+j=k} J_i J_j$. Then J_k is a T_2-ideal of $F\langle Y_t, Z_s\rangle$.

LEMMA 7. *There is $n \in \mathbf{N}$ such that $J_n \subseteq m(I_0)$. Also, modulo $J_{k+1} + m(I_0)$ the J_k is spanned by elements of the form*

$$f(v_1, \ldots, v_k)E, \tag{27}$$

where f is a multilinear monomial in $B\langle Y, Z\rangle$, $v_i \in F\langle Y_t, Z_s\rangle^2$, and $E \in \{1, R_a, R_a R_b\}$ $(a, b \in Y_t \cup Z_s)$.

Proof. Denote $A = F\langle Y_t, Z_s\rangle$ and let $I = m(I_0) \cap A$. By the super analog of (25)–(26), the algebra A/I satisfies the identities

$$J^*(x, y, z)S = 0, \tag{28}$$

where $S = R_{ab}$ or $S = R_a R_b$. We rewrite it in the operator form (modulo (21)):

$$\left(R_x R_y + (-1)^{\bar{x}\bar{y}} R_y R_x + R_{xy}\right)S = 0. \tag{29}$$

Let $C = \{a \in A \mid aA \cdot A + aA^2 \subseteq I\}$. Then C is a T_2-ideal of A and the graded algebra A/C is an N-superalgebra, where N is the variety of commutative algebras satisfying the identity $x^3 = 0$. The square of such an algebra is nilpotent [8]. Since the identity of nilpotency of any kind is fixed under the mapping $f \to f^*$ (18), A^2 is also nilpotent modulo C, $(A^2)^n = 0$. Also, by (29), the product of any two ideals in A/C is an ideal. Hence, $J_k = (A^2)^k$ modulo C. This yields $J_i J_j = (A^2)^i (A^2)^j$ modulo $m(I_0)$ and $J_n \subseteq m(I_0)$.

Finally, modulo $m(I_0) + J_{k+1}$ the vector space J_k is spanned by elements $f \cdot E$, where $f \in (A^2)^k$ and E is a product of operators R_a, $a \in F\langle Y_t, Z_s\rangle$. By (29), we may assume that E has the form (27). ∎

Put $V_0 = B$, and $V_k = J_k\,\omega(B) + m(I_0)$ $(k \geq 1)$. Hence, we have a finite chain $V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = m(I_0)$.

PROPOSITION 8. *There is a finitely generated subalgebra* $\Phi \subseteq \mathrm{End}_F B$, *such that*

(1) $m(I_k)$ *and* V_k *are* Φ-*modules*;

(2) *let* $\overline{V}_k = V_k/V_{k+1}$; *then the image of* Φ *in* $\mathrm{End}_F \overline{V}_k$ *is a commutative subalgebra and* $\overline{V}_k$ *is a finitely generated* Φ-*module*.

Proof. Define the algebra Φ. It is generated over F by $\omega(Y_t)$ and the following operators ($z \in Z_s$ and $r \in \mathbf{N}$, $r \leq n$):

$$\delta_r(z) = \sum_{x \in Y_t \cup Z_s} \Delta_x\left(xR_z^{2r}\right).$$

By Lemma 5, $m(I_k)$ and V_k are Φ-modules (since $\omega(z)$ is an odd element of a super commutative algebra, its square is equal to 0, hence, $\omega(xR_z^{2r}) = \omega(x)\omega(z)^{2r} = 0$ for $r \geq 1$).

Next, take a generator $f\omega(h)$ of V_k (modulo V_{k+1}) where f has the form (27) and $h \in B$. Also, $v_j = (a_j b_j)E_j$, where $a_j, b_j \in Y_t \cup Z_s$ and E_j is a product of operators R_c, $c \in Y_t \cup Z_s$. Thanks to (29), the operators in E_j super anticommute to each other, i.e., $R_c R_{c'} = -(-1)^{\bar{c}\bar{c}'} R_{c'} R_c$ (it is clear, if $k > 1$; in the case $k = 1$ we also may assume that either $E_i = 1$ or $E = R_a R_b$ and, therefore, the identity (29) works). In particular, the number of $c \in Y_t$ does not exceed t.

We shall prove that modulo V_{k+1}

$$f(v_1, \ldots, v_k)\,\delta_r(z) = \sum_{j=1}^{k} f|_{E_j = E_j R_z^{2r}}. \tag{30}$$

Indeed, by (29), modulo $I + (A^2)^2$ we have

$$(ab)\,\delta_r(z) = (ab)R_z^{2r}. \tag{31}$$

Also, in E_j the operator R_z^{2i} commutes with any operator R_x, $x \in Y_t \cup Z_s$, that proves (30).

In particular, $[\delta_r(z), \delta_{r'}(z')] = 0$ modulo V_{k+1}, hence, Φ acts on $\overline{V}_k$ as a commutative algebra.

Now let us show that $\overline{V}_k$ is a finitely generated Φ-module (cf. [1, Sect. 5]). Denote by t_j the operator of replacement of E_j with $E_j R_z^2$ in (27) (v_j is of the form $(a_j b_j)E_j$). Evidently, $t_1, \ldots, t_k$ are mutually commuting and $\delta_r(z)$ acts on (27) as a symmetric function $t_1^r + \cdots + t_k^r$. Since t_j is integral over the algebra generated by such functions, modulo V_{k+1} the element (27) belongs to the module over $\mathrm{alg}_F\{\delta_r(z) | r \leq k,\ z \in Z_s\}$ generated by elements of the same form (27) where in each E_j the number of operators R_z, for each $z \in Z_s$, is not bigger than $2(k-1)$. Also, as it has been noticed above, in E_j the operator R_{y_i} does occur twice for each $i = 1, \ldots, t$.

Finally, the total degree of h in Z_s is also not bigger than s. It means that the total degree of $f\omega(h)$ is restricted and, therefore, the number of generators is finite. The proposition is proved. ∎

Thus V_k/V_{k+1} is a Noetherian Φ-module, hence the chain $(m(I_i) \cap V_k)/V_{k+1}$, $i = 1, 2, \ldots,$ becomes stationary for any k, hence, $m(I_i)$, $i = 1, 2, \ldots,$ also does it. Therefore, by Lemma 6, Theorem 3 is also proved.

5. EXAMPLES

Let U be a vector space over F and let Z be a subspace of $\mathrm{End}_F U$. On the direct sum $U \dotplus Z$ define a multiplication letting $uz = u \cdot z$ for $u \in U$ and $z \in Z$; the other products are equal to zero. Then the identities (5) hold and, hence, by [18, p. 210], $B(U, Z) = eF \dotplus U \dotplus Z$, where e is an idempotent, is a Bernstein algebra.

THEOREM 5. *Let* $\mathrm{Ass}\langle X \rangle$ *be a free associative algebra with an infinite set of free generators* X, *let* U *be a free* $\mathrm{Ass}\langle X \rangle$*-module, and* $Z = \mathrm{vect}_F X$. *Then the variety of baric algebras* $\mathrm{var}(B(U, Z), \omega)$ *cannot be defined by identities of the Grassmann envelope of any baric superalgebra of finite rank* (*moreover, neither the number of even generators nor odd ones can be finite*).

Proof. Suppose that $T(B(U, Z), \omega) = T(G(A), \omega)$, where (A, ω) is a baric superalgebra generated by t even elements and s odd ones, where s or t is finite.

For any $a \in B(U, Z)$ denote $T_a = R_a - \omega(a)/2 \in \mathrm{End}_F B(U, Z)$. Notice that $UT_{ab} = 0$ for any $a, b \in B(U, Z)$. Hence, if I is an ideal of $F\langle X \rangle$ generated by associators (a, b, c), where $a, b, c \in F\langle X \rangle$, then $IT_{xy} \subseteq T(B(U, Z), \omega)$. Since $I = I^*$ by (18), the superalgebra B also satisfies any identity in IT_{xy}. Suppose that $s < \infty$ (the case $t < \infty$ is similar) and consider

$$f = (x, y, z)T_{x_1} \cdots T_{x_{t+1}}.$$

Then, obviously, B satisfies the identity

$$g = \sum_{\sigma \in S(s+1)} \mathrm{sgn}(\sigma) f(x, y, z, x_{\sigma(1)}, \ldots, x_{\sigma(s+1)}).$$

Hence, by (18),

$$g^* = \sum_{\sigma \in S(s+1)} f(x, y, z, x_{\sigma(1)}, \ldots, x_{\sigma(s+1)}),$$

is an identity of $(B(U, Z), \omega)$. On the other hand, replacing x with a free generator of U, y, z and x_i, $i = 1, \dots, s+1$ with R_y, R_z, R_{x_i}, respectively, we get a nonzero element. ∎

Now we will give an example of an exceptional Bernstein algebra which has no finite basis of identities.

THEOREM 6. *There is a sequence of Bernstein algebras* (A_i, ω), *where* (A_1, ω) *is finite dimensional and* $T(A_i, \omega) \subset T(A_{i+1}, \omega)$, $i \in \mathbf{N}$. *Moreover, the chain of ideals of identities* $T(A_i)$, $i \in \mathbf{N}$, *does not become stationary.*

The example will be based on an ascending chain of S-ideals of $\mathrm{Ass}\langle X\rangle$ which does not become stationary modulo identities of a finite dimensional algebra.

Let $\mathrm{UT}_3(F)$ be the algebra of upper triangular matrices of order 3. Then its ideal of identities I is generated by the polynomial $[x_1, x_2][x_3, x_4][x_5, x_6]$ [12]. Recall that an ideal I of a free algebra is an S-ideal if it is closed under replacing variables with linear combinations of them. Also, we call an S-ideal Γ of $\mathrm{Ass}\langle X\rangle$ unitary closed if for any $f \in \Gamma$, we have $f\Delta_x(1) \in \Gamma$ (cf. [20, Sect. 1]).

LEMMA 8. *Let* Γ_j *be the S-ideal of* $\mathrm{Ass}\langle X\rangle$ *generated by polynomials* $a \cdot b$, *where* a, b *are commutators in variables of equal degree* $\le j$. *Then* Γ_i, $i \in \mathbf{N}$, *is a strictly ascending modulo I chain of unitary closed S-ideals.*

Proof. Let $f = a_j^2$, where a_j is the commutator $[x, y, \dots, y] = [\dots[x, y], y, \dots, y]$ of degree $j+1$ ($a_0 = x$). Suppose that $f = g + h$ for some $g \in \Gamma_j$ and $h \in I$. Notice that the equality is homogeneous in each variable. Besides, the T-ideal I is generated by the polynomial $[x_1, x_2][x_3, x_4][x_5, x_6]$, hence, $h = 0$ in $\mathrm{Ass}\langle X\rangle$, i.e., $f = g$ in $\mathrm{Ass}\langle X\rangle$. Thus, $f = \sum_{i=0,\ k=0}^{j-1,\ n} \alpha_{i,k} a_i^2 \,\mathrm{ad}(y)^k y^{2(j-i)-k}$, where $\alpha_{i,k} \in F$ and $\mathrm{ad}(y)$: $a \to [a, y]$. Recall that the $\mathrm{Ass}\langle x, y\rangle$ is the universal envelope of the free Lie algebra $L = L\langle x, y\rangle$ and as a vector space it has a basis which includes the elements $a_p a_q y^r$, where $p \ge q \ge 0$, $r \ge 0$ and a basis of L [9, p. 160]. In particular, this yields $2(j-i) - k = 0$, i.e., $k = 2(j-i)$. Let r be maximal such that $\alpha_{r,k} \ne 0$. On the other hand, modulo L we have $a_i^2 \,\mathrm{ad}(y)^{2(j-i)} = \sum_{p=0}^{j-i} n_p a_{i+p} a_{2j-i-p}$, where $n_p \in \mathbf{N}$. Hence, f is a linear combination of elements of the indicated basis of $\mathrm{Ass}\langle x, y\rangle$, where the coefficient at $a_{2j-r} a_r$ is not zero, which is impossible ($r < j$). ∎

Proof of Theorem 6. Let J_k be the T-ideal of $F\langle X\rangle$ generated by the set

$$S_k = \left\{ (x_1, x_2, x_3) f\left(R_{x_4}, \dots, R_{x_n}\right) \middle| f \in \Gamma_k \right\}.$$

We shall show that the chain of T-ideals $J_k + T(A)$, where $A = B(U, Z)$, U is a 3-dimensional space, and $Z = UT_3(F)$, does not become stationary.

Take a multilinear $f = f(x_1, \ldots, x_n) \in \Gamma_{k+1} \setminus \Gamma_k$ and suppose that

$$g = (x_1, x_2, x_3) f\left(R_{x_4}, \ldots, R_{x_n}\right) \in J_k + T(A).$$

Then, modulo $T(A)$,

$$g = \sum_{a, b, c \in V\langle X \rangle, h} (a, b, c) \cdot h,$$

where $h = h_1(R_d, \ldots, R_w)$, $h_1 \in \Gamma_k$, and $d, \ldots, w \in V\langle X \rangle$.

We replace x_i with $\bar{x}_i$, where $\bar{x}_1 = u \in U$, $\bar{x}_2 = \bar{x}_3 = e$, and $\bar{x}_i = z_i \in Z$, $i > 3$. Observe that $(\bar{a}, \bar{b}, \bar{c}) \in U$, hence, if among $d, \ldots, w$ there is a monomial of degree > 1 then we get $(\bar{a}, \bar{b}, \bar{c})\bar{h}$ to be zero. Also, $(\bar{a}, \bar{b}, \bar{c}) = u \cdot E$, where E is the product of R_{z_j}. Since Γ_k is unitary closed, $(\bar{a}, \bar{b}, \bar{c})\bar{h}$ has the form $u \cdot p(R_{z_4}, \ldots, R_{z_n})$ for some $p \in \Gamma_k$. Summarizing, we get $u \cdot f(R_{z_4}, \ldots, R_{z_n}) = u \cdot q(R_{z_4}, \ldots, R_{z_n})$, where $q \in \Gamma_k$, for any $u \in U$ and $z_j \in Z$. Hence, $f - q$ is an identity of $\mathrm{UT}_3(F)$, i.e., $f \in \Gamma_k + I$, which is impossible. ∎

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